Thin Shells for Computer Graphics:
A short course

Adrian Secord
Media Research Lab, Department of Computer Science
New York University

December 10, 2006

Contents

Chapter 1

Introduction

This is intended to be a short course of the geometry, physics and computational methods used for thin shells in computer graphics. It is based on a depth-exam syllabus created with the help of Professor Denis Zorin in 2005.[1]

Chapter 2 on continuous concepts is complete, but Chapter 3 on the discrete analogs is specified as certain readings.

---

[1] NYU computer science has a "depth exam" instead of "comprehensive exams" used at many schools. The depth exam is partially a presentation of initial research and partially an exam on concepts related to one's proposed area of research. The material presented here relates to thin shells in computer graphics as an area of research.

Chapter 2

The continuous

2.1 Basic differential geometry

This section follows do Cormo's *Differential Geometry of Curves and Surfaces* [do Cormo, 1976] closely, but focusses on local properties of curves and surfaces.

2.1.1 Definitions for curves

We start with a reminder of some basic definitions. Most of these definitions will have equivalent concepts with respect to surfaces. A *parameterized curve* is a map from some interval of the real line to a subset of $\mathbb{R}^3$. The *trace* of a curve is its image in $\mathbb{R}^3$, and is simply a set of points. Note that many curves can have the same trace, for example, there are many ways to parameterize the unit circle. The map does not have to be one-to-one, that is, the curve can intersect itself. The term *differentiable* or *smooth* means infinitely differentiable, that is, of class C^∞.

An example of a curve is the map $\alpha : (a, b) \to \mathbb{R}^3$ where $\alpha(t) = (x(t), y(t), z(t))$. Since the map is smooth, we can define the *velocity* of the curve at a point t as $\alpha'(t) = (x'(t), y'(t), z'(t))$, where the prime indicates differentiation. The velocity is a vector quantity at every point and is also known as the *tangent vector.* The line containing the point on the curve and the tangent vector is the *tangent line.* A point on a curve with zero velocity has no tangent, and these points are called *singular points.* We generally want a tangent at every point, so we restrict ourselves to *regular curves*, that is, curves without singular points.

The distance measured along a curve is called *arc length* and it is defined by

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt,$$

where $|\cdot|$ denotes Euclidean distance in $\mathbb{R}^3$. It is possible to find a parameterization of a regular curve such that the parameter t is the arc length measured from some point. The velocity of such curves is one, which is quite useful in practice. We will assume that our curves are parameterized by arc length in the material that follows.

2.1.2 Parameterizing a curve by arc length

To parameterize a curve by arc length, the procedure is

1. Find the arc length $s(t)$. If the curve is regular then $s(t)$ is a monotonically increasing function.
2. Calculate the inverse of the arc length $t(s)$. This is another monotonically increasing function.
3. Set the new parameter $u = t(s)$. Now $\alpha(u)$ has arc length parameterization.

Intuitively, if the curve if "moving" quickly in some region, then its arc length will increase rapidly and the inverse $t(s)$ will increase proportionally slowly. In the parameterized curve $\alpha(u)$ these two factors will cancel out and the curve will increase in length uniformly. What is the velocity of $\alpha(u)$? We have

$$\begin{aligned}
\alpha'(u) &= \frac{d}{ds}\alpha(t(s)) && (2.1)\\
&= \alpha'(t)t'(s) && (2.2)\\
&= \alpha'(t)/s'(t) && (2.3)\\
&= \alpha'(t)/|\alpha'(t)|, && (2.4)
\end{aligned}$$

where we have used the fact that $\frac{d}{ds}t(s) = 1/\frac{d}{dt}s(t)$ when s and t are inverses of each other, and the definition of arc length in equation (2.4). Hence we have found that the velocity $|\alpha'(u)|$ is one for an arc length parameterized curve.

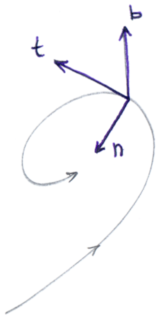

Figure 2.1: Curve with tangent t, normal n and binormal b.

2.1.3 Curvature of curves

Given a curve $\alpha(s)$ parameterized by arc length, we want to describe the bending and twisting of the curve at a point. Starting with the unit tangent vector $t(s) = \alpha'(s)$, we can examine the vector $t'(s) = \alpha''(s) = k(s)n(s)$. This is a vector which we break into two parts: a scalar *curvature* $k(s)$ and a vector *normal*. Hence the curvature is defined as $k(s) = |\alpha''(s)|$ and the normal is uniquely defined if $k(s) \neq 0$. The curvature describes how the curve pulls away from the tangent. The inverse of the curvature $R = 1/k$ is called the *radius of curvature* and describes the size of a circle with the same curvature. The plane that contains $t(s)$ and $n(s)$ is called the *osculating plane*. If a curve is completely contained in a plane, then the osculating plane at every point coincides with the containing plane. For this section, we will assume that $k(s) \neq 0$ so that the normal is defined everywhere.

The unit *binormal vector* $b(s) = t(s) \times n(s)$ is the second direction in which the curve can bend and is perpendicular to the osculating plane by definition[1]. The rate of change of $b(s)$ describes how the osculating planes of neighbouring points differ, or put another way, how the curve pulls out of the osculating plane.

It is useful to know how the three vectors change with changes of parameter. By definition, we have $t'(s) = k(s)n(s)$. For the binormal, we

---

[1] We denote the cross product of two vectors $u, v \in \mathbb{R}^3$ as $u \times v$.

have

$$b'(s) = t'(s) \times n(s) + t(s) \times n'(s) \quad (2.5)$$
$$= k(s)n(s) \times n(s) + t(s) \times n'(s) \quad (2.6)$$
$$= t(s) \times n'(s), \quad (2.7)$$

and it follows that $b'(s)$ is parallel to the normal.(TODO) Hence we can write $b'(s) = \tau(s)n(s)$, where $\tau(s)$ is the scalar *torsion*[2]. Finally, the derivative of the normal can be expressed in terms of the other quantities:

Explain why b'(s) is parallel to the normal.

$$n'(s) = b'(s) \times t(s) + b(s) \times t'(s) \quad (2.8)$$
$$= \tau(s)n(s) \times t(s) + b(s) \times k(s)n(s) \quad (2.9)$$
$$= -\tau(s)b(s) - k(s)t(s) \quad (2.10)$$

To summarize, the behaviour of a arc length parameterized curve can be characterized by three orthonormal vectors, namely the tangent $t(s)$, the normal $n(s)$ and the binormal $b(s)$, and two scalars, namely the curvature $k(s)$ and the torsion $\tau(s)$. The frame defined by the three vectors and the point $\alpha(s)$ is called the *Frenet frame*.

The *fundamental theorem of the local theory of curves* says that all curves with identical $k(s)$ and $\tau(s)$ are identical up to rigid transformations. That is, the curvature and torsion uniquely characterize the local behaviour of a curve.

2.1.4 Regular surfaces

A *regular surface* is defined using the following pieces:

- a subset S of $\mathbb{R}^3$ (the surface),
- an arbitrary point $p \in S$,
- a neighbourhood $V \subset \mathbb{R}^3$ of p,
- a subset $U \subset \mathbb{R}^2$, and
- a mapping x from U onto $V \cap S$.

The subset S is called a regular surface if

1. x is differentiable, that is, its component functions have continuous paritial derivatives of all orders,

---

[2] Some define the torsion as $b'(s) = -\tau(s)n(s)$.

2. x is a homeomorphism, that is, it is one-to-one and has a continuous inverse x^{-1}, and
3. the Jacobian of x is full rank.

The first condition simply guarantees that we can find derivative-related quantities on the surface such as tangent planes. The second condition forbids problematic features such as self-intersections, since it is again impossible to define tangent planes at these features. The existence and continuity of the inverse allows us to show that the various parameterizations available at a point are equivalent and not special. The third condition is called the *regularity condition* and ensures that coordinate lines in U do not collapse and become colinear in S.(TODO) The mapping x is called the *parameterization* of the surface.

A better explanation of the regularity condition.

2.1.5 Tangent plane and surface normal

In moving to the geometry of surfaces, many concepts are bootstrapped by examining the behaviour of curves within the surface, typically passing through some point of interest. For example, Appendix A uses curves to define the mapping of vectors from one space to another.

The tangent plane $T_p(S)$ at a point p on a regular surface S is defined as the subspace containing the tangents of all possible curves passing through p. By the first two conditions of a regular surface (smoothness and non–self-intersection, see Section 2.1.4), we are guaranteed that the curve tangents form a unique plane. Using the concept of a differential (Appendix A), the tangent plane can be defined as $dF_p(\mathbb{R}^2)$, the mapping of all possible tangent vectors at a point. Just as the differential does not depend on an underlying parameterization, neither does the definition of the tangent plane.

The *surface normal* is the unit vector perpendicular to all vectors in the tangent plane. There are two such vectors that satisfy the definition, and without a parameterization it is not possible to define a single normal vector, only a normal line. However, if a parameterization $x : (u, v) \to \mathbb{R}^3$ is given, then the partial derivatives x_u and x_v define a unique normal at a point p via the rule

$$N(p) = \frac{x_u \times x_v}{|x_u \times x_v|}(p). \tag{2.11}$$

Note that the regularity condition guarantees that the denominator does not vanish. The field of normal vectors on the surface is can be shown to be continuous if the surface is *orientable*. Orientation of surfaces is not discussed further in these notes.

2.1.6 First fundamental form

The inner product[3] of $\mathbb{R}^3$ generates a quadratic form[4] I_p that takes vectors in the tangent space to the real line, that is, $I_p : T_p(S) \to \mathbb{R}$. Given a vector w in the tangent plane, the *first fundamental form* is given by

$$I_p(w) = \langle w, w \rangle = |w|^2 \geq 0. \tag{2.12}$$

This simple function encodes distance, area and angle information in a convienient form.

Given a parameterization $x(u, v)$ of the surface, we can express the first fundamental form in the basis $\{x_u, x_v\}$. Recall that a tangent vector w is by definition the tangent of some curve $\alpha(t) = x(u(t), v(t))$ with $\alpha(0) = p$. Expanding the first fundamental form, we get

$$\begin{aligned} I_p(w) &= \langle \alpha'(0), \alpha'(0) \rangle & (2.13)\\ &= \langle x_u u' + x_v v', x_u u' + x_v v' \rangle & (2.14)\\ &= \langle x_u, x_u \rangle u'^{\,2} + 2\langle x_u, x_v \rangle u'v' + \langle x_v, x_v \rangle v'^{\,2} & (2.15)\\ &= Eu'^{\,2} + 2Fu'v' + Gv'^{\,2} & (2.16) \end{aligned}$$

where all quantities involving t are taken at $t = 0$. This expansion shows that given a parameterization, we can compute the *coefficients of the first fundamental form* $E(u, v)$, $F(u, v)$ and $G(u, v)$ to give a simple characterization of the first fundamental form at the point $p = x(u, v)$.

Define the *deformation* of the surface as

$$S = \nabla x = \begin{pmatrix} x_u^1 & x_v^1 \\ x_u^2 & x_v^2 \end{pmatrix}, \tag{2.17}$$

where $S : \mathbb{R}^2 \to \mathbb{R}^2$ and the parameterization $x(u, v) = (x^1(u, v), x^2(u, v))$. Then if $w = (w^1, w^2)$ in the parameter space, it is transformed to $Sw = x_u w^1 + x_v w^2$ in the tangent plane to the surface. The first fundamental form, then, becomes

$$I_p(w) = \langle Sw, Sw \rangle = w^T S^T S w. \tag{2.18}$$

This is simply a restatement of equation (2.16) in vector form. The coefficients can be extracted from $S^T S$.

---

[3] We denote the *inner* or *dot product* of two vectors $v, w \in \mathbb{R}^n$ by $\langle v, w \rangle$.

[4] A *quadratic form* in n variables is simply an expression $Q : \mathbb{R}^n \to \mathbb{R}$ of the form $Q(x) = x^T A x$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$.

The inner product allows us to compute metric quantities on the surface without having to directly measure things in $\mathbb{R}^3$. The standard uses of inner product from linear algebra carry over directly. The arc length s of a curve $\alpha(t)$ is given by

$$\begin{aligned} s(t) &= \int |\alpha'(t)| dt && (2.19) \\ &= \int \sqrt{I(\alpha'(t))} dt && (2.20) \\ &= \int \sqrt{Eu'^{\,2} + Fu'v' + Gv'^{\,2}} dt. && (2.21) \end{aligned}$$

The angle between two curves $\alpha(t)$, $\beta(t)$ intersecting at $t = t_0$ on the surface is given by

$$\cos\theta = \frac{\langle \alpha'(t_0), \beta'(t_0) \rangle}{|\alpha'(t_0)||\beta'(t_0)|}. \tag{2.22}$$

Recall that coordinate curves always have tangents x_u and x_v, so the angle between these curves is

$$\cos\phi = \frac{\langle x_u, x_v \rangle}{|x_u||x_v|} = \frac{F}{\sqrt{EG}} \tag{2.23}$$

by definition. Theses curves are othogonal when $F = 0$, and a parameterization where $F(u, v) \equiv 0$ is called an *orthogonal parameterization* or *conformal map.*

The area of a region R of a regular surface can be computed by mapping R back to into the parameter space to get $Q = x^{-1}(R)$ and integrating

$$A(R) = \iint_Q |x_u \times x_v| du\, dv \tag{2.24}$$

It can be show that this integral is independent of the parameterization x. Note the following:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && (2.25) \\ \left(\frac{|x_u \times x_v|}{|x_u||x_v|}\right)^2 + \left(\frac{\langle x_u, x_v \rangle}{|x_u||x_v|}\right)^2 &= 1 && (2.26) \\ |x_u \times x_v|^2 + \langle x_u, x_v \rangle^2 &= |x_u|^2 |x_v|^2 && (2.27) \\ |x_u \times x_v|^2 &= EG - F^2. && (2.28) \end{aligned}$$

In terms of the coefficients of the first fundamental form, the area becomes

$$A(R) = \iint_Q \sqrt{EG - F^2} du\, dv \tag{2.29}$$

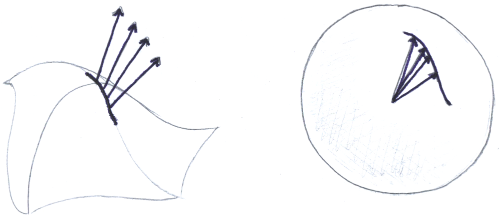

Figure 2.2: Surface and corresponding Gauss map along a curve.

2.1.7 Gauss map

Recall from Section 2.1.5 that given a parameterization x of a regular surface S, we can define a field of unique surface normals $N : S \to \mathbb{R}^3$

$$N(p) = \frac{x_u \times x_v}{|x_u \times x_v|}.$$

The space of all possible normals lies in the unit sphere in $\mathbb{R}^3$, so we can identify with each normal its two-dimensional location in the sphere. This map $N : S \to \mathbb{R}^2$ is called the *Gauss map.* The Gauss map is differentiable and the differential dN_p maps vectors in the tangent plane to vectors in the tangent plane. That is, given a point p on the surface and a direction v in its tangent plane, $dN_p(v)$ gives the change in surface normal as you move from p to $p + \epsilon v$. The change of the surface normal is a vector again in the tangent plane of S at p. Hence the differential dN_p gives the change in surface normal in the neighbourhood of p.

2.1.8 Normal curvature

Given a regular surface and a curve within that surface, the *normal curvature* at a point is the amount of the curve's curvature in the direction of the surface normal. The curve on the surface passes through a point p, with tangent v, curvature k and normal n. Given the surface normal N, the normal curvature k_n is the length of the projection of kn onto N, namely $\langle kn, N \rangle$.

If we cut a plane containing both N and v through the surface, we generate a second curve on the surface that agrees with the first curve at p. The normal curvature is the same as the curvature of this second curve at p. In addition, the *Meusnier theorem* states that the curves generated by cutting any plane that contains both p and v all have the same normal curvature.

(TODO) **Diagram**

2.1.9 Second fundamental form (shape operator)

The second fundamental form is similar to the first fundamental form (Section 2.1.6), except that it relates a vector v in the tangent plane to the change in normal in the direction of v. Specifically, the *second fundamental form* or *shape operator* at a point p is defined as

$$II_p(v) = -\langle dN_p(v), v\rangle. \tag{2.30}$$

Recall that $dN_p(v)$ is the change in surface normal in the direction of v at a point p. Similarly to the case of curves (Section 2.1.3), the change in surface normals encodes information about the curvature of the surface.

To see how this might be true, consider a regular arc length-parameterized curve $\alpha(s)$ on the surface. The field of normals restricted to the curve is also a function of s: $N(s) = N \circ \alpha(s)$. Since the normal is always perpendicular to the tangent plane, we have $\langle N(s), \alpha'(s)\rangle = 0$, which we can differentiate to obtain $\langle N'(s), \alpha'(s)\rangle + \langle N(s), \alpha''(s)\rangle = 0$. Therefore

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle && (2.31)\\ &= -\langle N'(0), \alpha'(0)\rangle && (2.32)\\ &= \langle N(0), \alpha''(0)\rangle && (2.33)\\ &= \langle N, kn\rangle(p) && (2.34)\\ &= k_n(p) && (2.35) \end{aligned}$$

So along a curve on the surface, the second fundamental form at a point gives the normal curvature to an arc length-parameterized curve.

2.1.10 Principal curvatures

The maximum and minimum normal curvatures at a point p on a regular surface are called the *principal curvatures* at p, usually denoted k_1 and k_2, respectively. The corresponding directions, e_1 and e_2, are called the *principal directions* at p.

The principal curvatures and directions have an inuitive interpretation in terms of the second fundamental form (Section 2.1.9) — the principal directions are its eigenvectors[5]. The second fundamental form (and thus the normal curvatures) takes its minimum and maximum values at the eigenvectors, and those values are the negative of the eigenvalues. (The negative sign comes from the definition of the second fundamental form: $II_p(v) = -\langle dN_p(v), v\rangle$.)

The principal directions form a convienient vector basis for determining curvatures of any direction in the tangent plane. If v is an arbitrary unit vector in the tangent plane, then clearly $v = e_1 \cos\theta + e_2 \sin\theta$, where θ is the angle between v and e_1. Working from the definition of the second fundamental form, we have

$$
\begin{aligned}
k_n &= II_p(v) && (2.36)\\
&= -\langle dN_p(v), v\rangle && (2.37)\\
&= -\langle dN_p(e_1\cos\theta + e_2\sin\theta), e_1\cos\theta + e_2\sin\theta\rangle && (2.38)\\
&= \langle e_1 k_1\cos\theta + e_2 k_2 \sin\theta, e_1\cos\theta + e_2\sin\theta\rangle && (2.39)\\
&= k_1\cos^2\theta + k_2\sin^2\theta. && (2.40)
\end{aligned}
$$

The last expression is the *Euler formula*, and is the expression of the second fundamental form in the basis $\{e_1, e_2\}$.

2.1.11 Mean and Guassian curvatures

Several traditional measures of curvature are encoded in the differential of the Gauss map. *Gaussian curvature* is defined as the determinant of dN_p: $K = k_1k_2$. The *mean curvature* is defined as the negative of half of the trace of dN_p: $H = (k_1 + k_2)/2$.

2.1.12 Classification of surface points

The Gauss map at a point can be used to classify the local geometry by archetypal geometries. A point is called

- *elliptic* if $det(dN_p) > 0$ (e.g. a sphere),
- *hyperbolic* if $det(dN_p) < 0$ (e.g. a saddle),
- *parabolic* if $det(dN_p) = 0$ and $dN_p \neq 0$ (e.g. a cylinder), and
- *planar* if $det(dN_p) = 0$.

---

[5] Recall that if some operator A maps vectors to vectors of the same dimension, then v is called an *eigenvector* if $Av = \lambda v$, and λ is called its *eigenvalue* of dN_p. Note that this definition applies for any mapping of vectors to vectors, not just matrix operators.

2.1.13 Expressions for curvature of parametric surfaces

If we define a curve $\alpha(t) = (u(t), v(t))$ in parameter space, (and thus $(x(u(t)), x(v(t))$ on the surface) and restrict the normal field to the curve, then we get a function of normals along the curve $N(t)$. The derivative $N'(t) = N_u u' + N_v v'$ is precisely $dN(v)$, where $v = \alpha'(t)$. Futhermore, given a parameterization of the surface x, we have a basis for vectors in the tangent plane $\{x_u, x_v\}$ and N_u, N_v can be represented in this basis:

$$N_u = a_{11}x_u + a_{21}x_v \tag{2.41}$$
$$N_v = a_{12}x_u + a_{22}x_v \tag{2.42}$$

Labouring on, we have

$$\begin{aligned} N'(t) &= (a_{11}x_u + a_{21}x_v)u' + (a_{12}x_u + a_{22}x_v)v' && (2.43)\\ &= (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}x_v)v', && (2.44)\end{aligned}$$

or in vector form,

$$dN\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}. \tag{2.45}$$

So in the basis $\{x_u, x_v\}$, dN is the linear map given by the a_{ij}. This is convienient, and we now want to find expressions for the a_{ij} in terms of the parameterization x and its derivatives.

The shape operator can be expressed in terms of the parameterization by invoking it on the tangent to a curve $\alpha(t)$:

$$\begin{aligned} -\langle dN_p(\alpha'(t)), \alpha'(t)\rangle &= \langle N'(t), x_u u' + x_v v'\rangle && (2.46)\\ &= -\langle N_u u' + N_v v', x_u u' + x_v v'\rangle && (2.47)\\ &= -\langle N_u, x_u\rangle u'^{\,2} - \langle N_u, x_v\rangle u'v' - \langle N_v, x_u\rangle u'v' - \langle N_v, x_v\rangle v'^{\,2} && (2.48)\\ &= eu'^{\,2} + 2fu'v' + gv'^{\,2}, && (2.49)\end{aligned}$$

where we have defined

$$\begin{aligned} e &= -\langle N_u, x_u\rangle = \langle N, x_{uu}\rangle && (2.50)\\ f &= -\langle N_u, x_v\rangle = \langle N, x_{uv}\rangle = \langle N, x_{vu}\rangle = -\langle N_v, x_u\rangle && (2.51)\\ g &= -\langle N_v, x_v\rangle = \langle N, x_{vv}\rangle. && (2.52)\end{aligned}$$

The transformations producing the second derivatives of x are generated by noticing that since, say, x_u is in the tangent plane, it must be true that $\langle N, x_u\rangle = 0$, hence $\langle N_u, x_u\rangle + \langle N, x_{uu}\rangle = 0$.

Substituting (2.41)-(2.42) into (2.50)-(2.51) generates a set of relations between a_{ij}, $\{e, f, g\}$, and $\{E, F, G\}$, for example, $-e = \langle N_u, x_u \rangle = a_{11}E + a_{21}F$. When solved, these give

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} . \tag{2.53}$$

Hence, given a parameterization x of a surface, we can write down the differential or shape operator in the basis $\{x_u, x_v\}$.

2.1.14 Gauss-Bonnet theorem

2.1.15 One-forms

2.1.16 Closed forms

2.1.17 Stokes theorem

2.2 Deformable surface models

2.2.1 Linear elasticity

The linear elasticity of deformable materials is a classic subject. This section draws upon the excellent introduct by Feynman in [Feynman et al., 1989], Biot's useful [Biot, 1965] and Gould's clean and useful exposition in [Gould, 1994].

Stress

Stress is the description of forces inside or on the surface of a body. The stress at a point in the body will change depending on which plane through the point one examines. Feynman uses the device of imagining that the body is cut with the plane – the body will deform, which means that there are stresses across the plane before it is cut. Stress is a vector function of both position and orientation throughout the body. If we examine the three planes that are perpendicular to coordinate axes at a point, then it is clear that nine quantities are needed: one vector in $\mathbb{R}^3$for each plane. Figure 2.3 shows the stresses on three of six sides of a small cube of material. If we assume the cube is infinitesimally small, then stresses on the other faces will simply be the negatives of those shown. We label the components of stress as σ_{ij}, where σ_i is the stress vector in the ith direction. Note that, in general, σ_i does not lie parallel to the ith direction. A force-balancing argument can show that given the stresses in each of the coordinate directions, the stress across a plane with normal n is simply $\sigma_{ji} n_j$.

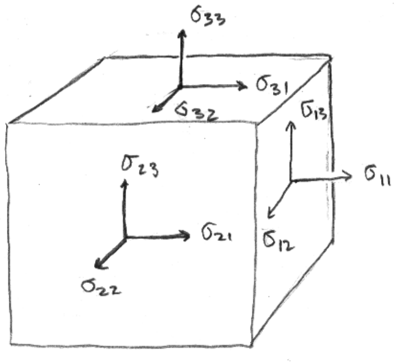

Figure 2.3: Differential cube with components of stress on three of six faces.

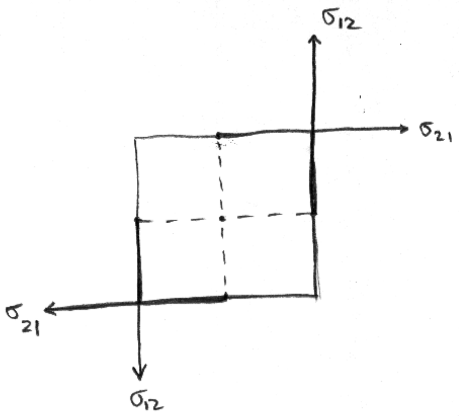

Figure 2.4: Torques on a unit cube. For illustration, stresses in the z direction and stresses perpendicular to the faces are ignored, since they do not change the torque in question.

We can also show that the stress tensor is symmetric by observing the torques induced on the centre of the cube by the components of stress. Figure 2.4 shows the situation schematically. The magnitude of the total torque induced on the centre is $\frac{1}{2}\sigma_{12} - \frac{1}{2}\sigma_{21} + \frac{1}{2}\sigma_{12} - \frac{1}{2}\sigma_{21} = \sigma_{12} - \sigma_{21}$. There cannot be any torques on the cube, otherwise it would start spinning. Hence $\sigma_{12} = \sigma_{21}$, and by extension the tensor σ is symmetric.

Strain

Strain is a geometric description of how each point in a body has deformed relative to the *undeformed state*[6].

We track a point x in the undeformed body to its corresponding point $\tilde{x}$ in the deformed body via a deformation $u(x)$: $\tilde{x} = x + u(x)$. We denote the distance between two nearby points in the undeformed state as $dx = (dx_1, dx_2, dx_3)$ and their images in the deformed state as $\tilde{dx} = (\tilde{dx}_1, \tilde{dx}_2, \tilde{dx}_3)$. The quantity we wish to examine is the difference in these distances squared:

$$|\tilde{dx}|^2 - |dx|^2 = \tilde{dx}_i\tilde{dx}_i - dx_i dx_i. \tag{2.54}$$

If we take the mapping between undeformed and deformed to be $\tilde{x}(x) = \tilde{x}(x_1, x_2, x_3)$, then we can express the differential quantities in the deformed state in terms of the undeformed variables. That is,

$$\begin{aligned} \tilde{dx}_i &= \frac{\partial \tilde{x}_i}{\partial x_1}dx_1 + \frac{\partial \tilde{x}_i}{\partial x_2}dx_2 + \frac{\partial \tilde{x}_i}{\partial x_3}dx_3 && (2.55) \\ &= \tilde{x}_{i,j}dx_j. && (2.56) \end{aligned}$$

The difference in squared distances becomes

$$\begin{aligned} |\tilde{dx}|^2 - |dx|^2 &= \tilde{dx}_i\tilde{dx}_i - dx_i dx_i && (2.57) \\ &= \tilde{x}_{i,j}dx_j\tilde{x}_{i,k}dx_k - dx_i dx_i && (2.58) \\ &= (\tilde{x}_{i,j}\tilde{x}_{i,k} - \delta_{ij}\delta_{ik})dx_j dx_k && (2.59) \\ &= ((x_i + u_i)_{,j}(x_i + u_i)_{,k} - \delta_{jk})dx_j dx_k && (2.60) \\ &= ((\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) - \delta_{jk})dx_j dx_k && (2.61) \\ &= (u_{j,k} + u_{k,j} + u_{i,j}u_{i,k})dx_j dx_k. && (2.62) \end{aligned}$$

---

[6] The undeformed state is sometimes called the *rest state*, but this is misleading, as the undeformed state does not have to be at rest – it could be moving uniformly as a rigid body.

Interchanging the indices i and k, we finally get to the definition of the components of strain:

$$\begin{aligned} |\tilde{dx}|^2 - |dx|^2 &= (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})dx_j dx_k && (2.63) \\ &= 2\varepsilon_{ij} dx_j dx_k && (2.64) \end{aligned}$$

where the components of strain are defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \tag{2.65}$$

The components of the gradient of the strain are often assumed to be small, allowing us to drop the last term of the strain:

$$\varepsilon_{ij} \approx \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.66}$$

Note that it is the *derivatives* of the strain that are assumed to be small, not the strains themselves. Thus large deformations can be described with this definition of strain if the strain varies smoothly throughout a body. Strain is a symmetric tensor of second order.

Material (stress-strain) relations

We have described stress, the forces in a body, and strain, a measure of geometric distortion of a body. What remains of linear elasticity is to describe the relationship between the two, which is a function of the material properties of a body.

In the one-dimensional case, Hooke's law states that stress is linearly dependent on strain: $\sigma = E\varepsilon$, where the constant E is the *modulus of elasticity* or *Young's modulus*. Even in the one-dimensional case, this is a major simplification. Figure XXX shows a measured stress-strain curve reproduced from [Gould, 1994]. (TODO) The term *elasticity* means that if a material is strained and then released, it returns along the same path in the stress-strain diagram. The path taken is not necessarily a straight line, but if it is, the term *linear elasticity* applies. It is remarkable that many materials do exhibit linear elasticity for some range of strains. However, all materials have a limit to the amount of strain they can react to linearly. As strain is increased, the *proportional limit* marks the start of non-linear behaviour where the slope decreases, finally flattening to zero at the *plastic limit*. Once the plastic limit is reached, further strain does not increase the stresses inside the material at all, and the material deforms permanently. If

Stress-strain figure

the material is released after the strain has exceeded the plastic limit, it will return along a different path in the stress-strain diagram than the initial elastic path, and its final stress-free state will have some non-zero strain (deformation).

Feynman [Feynman et al., 1989] explains that plastic deformation is a result of layers of molecules in the material slipping past each other and out of their initial formation. As long as this slippage does not occur, the intermolecular forces induce the elastic behaviour of the material as a whole, but once slippage has ocurred, the material is permanently deformed. Once the plastic deformation has occurred, then, the initial undeformed configuration has changed and the stress-strain diagram should realistically be given a new origin. Modelling plasticity as a change in the undeformed state is a reasonable approximiation for computation in particular.

The proceeding discussion is for the one-dimensional case. Similar things can be expected for a three-dimensional solid when examined along a single axis, but in general the situation is unsurprisingly more complicated. For the rest of our discussion we will be assuming that materials are linearly elastic. This is the most common assumption in computer graphics.

It is assumed that the stresses are governed by a strain energy density W that is quadratic in the strains:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = E_{ijkl}\varepsilon_{kl} \tag{2.67}$$

Clearly E_{ijkl} is a fourth-order tensor that mixes the components of ε to get a component of σ. E starts with $3^4 = 81$ components, but this is rapidly reduced by the following points and assumptions:

1. Both the stress and strain are symmetric, and we want to be able to invert the stress-strain relationship (get strains from stresses), hence E is symmetric. (Alternatively, W is continuous, hence E is symmetric.) This material is the most general considered, called *anisotropic* and E has 36 coefficients.

2. The material behaviour has one plane of symmetry. Having a plane of symmetry means that a stress applied normal or parallel to the plane induces displacements only in directions normal and/or parallel to the plane. This is called *monoclinic* and E has 13 coefficients.

3. The material behaviour has two planes of symmetry. This is called *orthotropic* and E has 9 coefficients.

4. The material has only three directions of independence. This is called *cubic* and E has three coefficients. The vast majority of engineering materials fall into this class.

5. The material has no directional dependence at all. This is called *isotropic* and E has only two coefficients. This is a commonly-assumed model of materials.

Without derivation, the components of E for an isotropic material take the form

$$\varepsilon_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}, \tag{2.68}$$

where μ and λ are called the Lamé constants.

Engineering constants

Many different forms of constants are used to describe the two degrees of freedom in an isotropic material. For reference, the conversions are:

- Lamé constants:

$$\mu = \frac{E}{2(1+\nu)} \tag{2.69}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{2.70}$$

- Young's modulus:

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda} \tag{2.71}$$

- Poisson's ratio:

$$\nu = \frac{\lambda}{2(\mu+\lambda)}, -1 < \nu < \frac{1}{2} \tag{2.72}$$

Chapter 3

Further reading

Some concepts and papers for further reading:

1. Physical models
    (a) Kirchhoff thin plate model
    (b) Biharmonic equation
    (c) Koiter's model
    (d) D. Braess *Finite Elements* [Braess, 2001]
2. Reparametization
    (a) Balmelli et al. *Space Optimized Texture Maps* [Balmelli et al., 2002]
    (b) Sander et al. *Signal-Specialized Parametrization* [Sander et al., 2002]
    (c) Alliez et al. remeshing work [Alliez, 2006]
3. Discrete geometry
    (a) Discretization of mean curvature
    (b) Normal cycles
    (c) Convergence properties of mean curvature discretization
    (d) Polthier Habilitation [Polthier, 2002]
    (e) Cohen-Steiner and Morvan paper [Cohen-Steiner and Morvan, 2003]
    (f) Discrete shell variations
        - Baraff and Witkin's *Large steps in colth simulation* [Baraff and Witkin, 1998]
        - Grinspun et al.'s *Discrete shells* [Grinspun et al., 2003]

  - Bridson et al.'s *Robust treatment of collisions, contact and friction for cloth animation* [Bridson et al., 2002]

4. Cloth and more generally deformable object simulation: a survey of methods with focus on bending.
   (a) Terzopoulos: [Terzopoulos and Fleischer, 1988, Qin and Terzopoulos, 1996]
   (b) Volino et al.'s *Versatile and efficient techniques for simulating cloth and other deformable objects* [Volino et al., 1995]
   (c) Choi and Ko's *Stable but responsive cloth* [Choi and Ko, 2002]
   (d) D Breen et al.'s *Predicting the drape of woven cloth using interacting particles* [Breen et al., 1994]

5. Adaptive cloth simulation
   (a) Molino et al.'s *A virtual node algorithm for changing mesh topology during simulation* [Molino et al., 2004]
   (b) Villard and Borouchaki's *Adaptive meshing for cloth animation* [Villard and Borouchaki, 2002]
   (c) Volkov and Li's *Real-time refinement and simplification of adaptive triangular meshes* [Volkov and Li, 2003]

6. Overview of the finite element method for plates and shells
   (a) Zienkiewicz, *Finite Element Method* [Zienkiewicz and Taylor, 1989]

Appendix A

Differential of a map

This section explains the *differential* (or derivative) of a map between some pair of spaces R^n and R^m. Here we use $\mathbb{R}^2$ and $\mathbb{R}^3$, but everything is clearly generalizable.

Given a map F from some connected subset U of $\mathbb{R}^2$ to a subset V of $\mathbb{R}^3$, the *differential* dF_p at a point $p \in U$ maps vectors in U to vectors in V. In particular, $v = dF_p(w)$ is the mapping of the vector w into $\mathbb{R}^3$. Since F is defined as a mapping of points and has no notion of vectors, the differential is defined by examining a curve $\alpha(t)$ passing through p and having velocity at p equal to w, that is, $\alpha(0) = p$ and $\alpha'(0) = w$.

The curve $\alpha(t)$ gets mapped to the curve $\beta(t) = F \circ \alpha(t)$ and v is defined as $\beta'(0)$. In the canonical basis, it can be shown that $\beta'(0) = J_p\alpha'(0)$ or $v = J_p w$, where J_p is the Jacobian matrix of partial derivatives of F at p. Hence the differential map is linear and does not depend on the actual curve $\alpha(t)$.

The Jacobian representation is convienient, since many standard results from calculus applied to maps result in simple matrix manipulations. The differential of a composition of two maps F and G ends up being the product of their respective Jacobians. Similarly, if the Jacobian of a map is invertible then the inverse function theorem applies and one can then talk of the inverse of a map F^{-1}.

Bibliography

[Alliez, 2006] Alliez, P. (2006). Pierre alliez's homepage. `http://www-sop.inria.fr/geometrica/team/Pierre.Alliez/`. Retrieved December 10, 2006. 20

[Balmelli et al., 2002] Balmelli, L., Taubin, G., and Bernardini, F. (2002). Space-optimized texture maps. *Computer Graphics Forum*, 21(3):411–411. 20

[Baraff and Witkin, 1998] Baraff, D. and Witkin, A. (1998). Large steps in cloth simulation. In Cohen, M., editor, *Proceedings of SIGGRAPH 98*, Annual Conference Series, Addison Wesley, pages 43–54. 20

[Biot, 1965] Biot, M. A. (1965). *Mechanics of Incremental Deformations.* John Wiley and Sons, Inc. 14

[Braess, 2001] Braess, D. (2001). *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics.* Cambridge University Press, Cambridge. Second Edition. 20

[Breen et al., 1994] Breen, D. E., House, D. H., and Wozny, M. J. (1994). Predicting the drape of woven cloth using interacting particles. In Glassner, A., editor, *Proceedings of SIGGRAPH '94 (Orlando, Florida, July 24–29, 1994)*, Computer Graphics Proceedings, Annual Conference Series, pages 365–372. ACM SIGGRAPH, ACM Press. 21

[Bridson et al., 2002] Bridson, R., Fedkiw, R., and Anderson, J. (2002). Robust treatment of collisions, contact and friction for cloth animation. *ACM Transactions on Graphics*, 21(3):594–603. 21

[Choi and Ko, 2002] Choi, K.-J. and Ko, H.-S. (2002). Stable but responsive cloth. *ACM Transactions on Graphics*, 21(3):604–611. 21

[Cohen-Steiner and Morvan, 2003] Cohen-Steiner, D. and Morvan, J.-M. (2003). Restricted delaunay triangulations and normal cycle. In *Proceedings of the nineteenth Conference on Computational Geometry (SCG-03)*, pages 312–321, New York. ACM Press. 20

[do Cormo, 1976] do Cormo, M. P. (1976). *Differential Geometry of Curves and Surfaces*. Prentice-Hall, Inc. 3

[Feynman et al., 1989] Feynman, R., Leighton, R. B., and Sands, M. L. (1989). *The Feynman Lectures on Physics*. Addison-Wesley Publishing Company. 14, 18

[Gould, 1994] Gould, P. L. (1994). *Introduction to Linear Elasticity, 2nd ed.* Springer-Verlag New York, Inc. 14, 17

[Grinspun et al., 2003] Grinspun, E., Hirani, A., Desbrun, M., and Schröder, P. (2003). Discrete shells. In Breen, D. and Lin, M., editors, *Proceedings of the 2003 ACM SIGGRAPH/Eurographics Symposium on Computer Animation (SCA-03)*, pages 62–67, Aire-la-Ville. Eurographics Association. 20

[Molino et al., 2004] Molino, N., Bao, Z., and Fedkiw, R. (2004). A virtual node algorithm for changing mesh topology during simulation. *ACM Transactions on Graphics*, 23(3):385–392. 21

[Polthier, 2002] Polthier, K. (2002). Polyhedral surfaces of constant mean curvature. Habilitationsschrift, Technische Universitt Berlin. 20

[Qin and Terzopoulos, 1996] Qin, H. and Terzopoulos, D. (1996). D-NURBS: A physics-Based framework for geometric design. *IEEE Transactions on Visualization and Computer Graphics*, 2(1):85–96. ISSN 1077-2626. 21

[Sander et al., 2002] Sander, P. V., Gortler, S. J., Snyder, J., and Hoppe, H. (2002). Signal-specialized parametrization. In Gibson, S. and Debevec, P. E., editors, *Rendering Techniques*, pages 87–98. Eurographics Association. 20

[Terzopoulos and Fleischer, 1988] Terzopoulos, D. and Fleischer, K. (1988). Modeling inelastic deformation: Viscoelasticity, plasticity, fracture. In Dill, J., editor, *Computer Graphics (SIGGRAPH ’88 Proceedings)*, volume 22, pages 269–278. 21

[Villard and Borouchaki, 2002] Villard, J. and Borouchaki, H. (2002). Adaptive meshing for cloth animation. In *Proceedings of the 11th International Meshing Roundtable (IMR 2002), September 15-18, 2002, Ithaca, New York, USA*, pages 243–252. 21

[Volino et al., 1995] Volino, P., Courchesne, M., and Thalmann, N. M. (1995). Versatile and efficient techniques for simulating cloth and other deformable objects. *Computer Graphics*, 29(Annual Conference Series):137–144. 21

[Volkov and Li, 2003] Volkov, V. and Li, L. (2003). Real-time refinement and simplification of adaptive triangular meshes. In van Wijk, G. T. J. J. and Moorhead, R. J., editors, *Proceedings of IEEE Visualization 2003*, pages 155–162. IEEE Computer Society, IEEE Computer Society Press. 21

[Zienkiewicz and Taylor, 1989] Zienkiewicz, O. C. and Taylor, R. (1989). *The finite element method*. McGraw-Hill. 21