Appendix: Proofs of statements

Proof of Proposition 2. First, consider a stationary point \( p \) of \( g \). As shown [Montgomery and Zippin 1955], there is a neighborhood \( U \) of \( p \) and a choice of smooth coordinates \( h : U \to \mathbb{R}^2 \) system on \( U \) such that \( g \) in these coordinates is a linear transformation \( A_p^g \), i.e. \( g = h^{-1} \circ A_p \circ h \). It follows that \( Dg(p) \) has the form \( V(p)A_p^gV(p)^{-1} \) where \( V(p) \) is the differential of the transformation \( h \) at point \( p \). As \( Dg(p)^2 = I \) at a stationary point, it follows that \( (A_p^g)^2 = I \). All such matrices have two eigenvalues, and both its eigenvalues satisfy \( \lambda^2 = 1 \).

Orientation-preserving \( g \). In this case, we show that \( g \) cannot be a reflection. In this case, both eigenvalues are either 1 or -1. Consider the set \( M^1(g) \) of all stationary points \( p \) with both eigenvalues of \( A_p^g \) equal to 1, and let \( M^2(g) \) be the set of all stationary points with both eigenvalues equal to -1. For points from \( M^1(g) \), \( A_p^g = I \), and \( g = h^{-1} \circ h \) is identity on \( U \), i.e. any stationary point of this type has an open neighborhood of stationary points of the same type. We conclude that \( M^1(g) \) is open. At any stationary point \( p \) from \( M^2(g) \), \( A_p^g = -I \), i.e. \( g \) has a single stationary point in \( U \) (itself). \( M^2(g) \) consists of isolated points. On the other hand, the set of all stationary points \( M(g) = M^1(g) \cup M^2(g) \) is closed, as the limit of any sequence of stationary points is stationary by continuity of \( g \). The limit of a sequence of points from \( M^1(g) \) has to be a point from \( M^2(g) \), as all points in \( M^2(g) \) are isolated, so the limit of points in \( M^1(g) \) is also in \( M^2(g) \). We conclude that \( M^1(g) \) is both open and closed. As we consider connected surfaces, an open/closed subset of an open surface has to be either empty or the whole surface. In the former case, \( M(g) = M^2(g) \), i.e. the stationary set consists of isolated points. A set of isolated points cannot separate the nonstationary subset into two disconnected components, so we conclude that this case is not possible for generalized reflections. In the latter case \( M(g) = M^1(g) \) is the whole surface, the map \( g \) is an identity, i.e. this case is not possible for reflections either.

Orientation-reversing \( g \). If \( g \) is orientation-reversing, at every stationary point, its differential \( Dg \) and linear form \( A \) has eigenvalues 1 and -1, and in \( h(U) \) the stationary set of \( A \) is a line \( \ell \), corresponding to the stationary curve \( h^{-1}(\ell) \) of \( g \). As this holds for any stationary point, the stationary curve can be extended indefinitely to an embedding of the real line or a circle in \( M \), forming a connected component of the stationary set. As the stationary set is closed, its connected components are also closed. But an embedding of a real line in a compact manifold cannot be closed; we conclude that the stationary set consists of embeddings of circles.

Consider a point \( p \) in one of the connected components \( M_1 \) of the non-stationary set \( M^c(g) \) of \( M \), mapped to a component \( M_2 \). Consider the set of all points in \( M_1 \) mapped to \( M_2 \), i.e. \( M_2 \cap g^{-1}(M_1) \). As \( M_2 \) is both open and closed in \( M^c(g) \), so is \( g^{-1}(M_2) \) by continuity of \( g \). Thus, \( M_2 \cap g^{-1}(M_2) \) is also open and closed, so it has to coincide with all of \( M_1 \) as \( M_1 \) is connected, i.e. \( g(M_1) \subset M_2 \). As \( g(p) \) is a point, by a similar argument, \( g(M_2) \subset M_1 \), so \( M_2 \) and \( M_1 \) are mapped to each other, and \( g(M_1) = M_2 \). Consider a point \( p \) on the boundary of \( M_1 \). As locally \( g \) acts as a linear reflection, mapping one part of the neighborhood \( U \) of \( p \) to the other, \( U \) has to consist of two disconnected parts from \( M_1 \) and \( M_2 \), i.e., any point on the boundary of \( M_1 \) separates it from \( M_2 \). Then the union of \( M_1 \) and \( M_2 \) and their boundary is closed in \( M \) and has no boundary, i.e., it has to coincide with \( M \).

Proof of Lemma 1. By Proposition 2, the differential \( Dg(p) \) at a stationary point \( p \) has two eigenvalues \( -1 \) and \( 1 \) (see proof above). Let \( e_1 \) be the eigenvector corresponding to eigenvalue \( 1 \); \( e_1 \) is a stationary direction of \( Dg(p) \). Now let us assume \( g \) is a change of coordinate system on \( T_p \) that aligns the first coordinate axis to \( e_1 \). If we express \( Dg \) with respect to the new frame, it must necessarily have the form:

\[
\begin{bmatrix}
1 & c \\
0 & d
\end{bmatrix}
\]

Since \( \det Dg(p) = -1 \) we necessarily have \( d = -1 \).

Proof of Corollary 3. Let \( g : M \to M \) be a diffeomorphism such that \( g^2 = Id \). As the stationary set partitions \( M \) into two connected domains, each has to be a disk, and so the curve is a topological circle (as it bounds a disk). Let \( b : M \to S \) be a one-to-one mapping from the surface to a sphere. Let \( \phi : S \to S \) be a homeomorphism of the sphere to itself that maps the stationary set of \( b \circ g \circ b^{-1} \) to a great circle. It follows that \( \phi \circ b \circ g \circ b^{-1} \circ \phi^{-1} \) has the circle as the stationary line. There is a stereographic projection \( P \) from the sphere to the plane mapping this circle to a line, say the \( x \)-axis. Let \( h = P \circ \phi \circ b \circ g \circ b^{-1} \circ \phi^{-1} \circ P^{-1} \), this is a homeomorphism from \( \mathbb{C} \) to \( S \) such that the \( x \)-axis is stationary, and it swaps two halves of the plane. Clearly, \( h^2 = Id \). Let \( R \) be the reflection of the plane that maps \( y \to -y \). Then \( R \circ h \) is a homeomorphism that maps each half-plane to itself. Let \( H_1 \) and \( H_2 \) be the two half-planes. Define the coordinate change \( f \) on the plane as \( f(x) = 2Id \circ R \circ R \circ h \). Then for \( x \in H_1 \), \( h(x) = h \circ R \circ R \circ Id = h^{-1} \circ R^{-1} \circ R \circ Id = (Rh)^{-1} \circ R \circ Id = f^{-1} \circ R \circ f \), and for \( x \in H_2 \), again, \( h(x) = Id \circ R \circ R \circ h = f^{-1} \circ R \circ f \), in other words, we got the factorization we wanted.

Proof of Lemma 4. Using the expression for \( R^2 \), we observe that it defines an analytic dependence of \( R^2 \) on \( Dg \), unless \( \det(Dg) = Dg^2 - Tr(Dg)I = 0 \), which, as can be seen by direct calculation, only happens if \( Dg \) is a similarity transformation. However, as \( Dg \) is orientation-reversing, this is not possible. Since \( g^2 = Id \) then \( Dg \circ g \circ Dg = Id \). Since at a point \( p \), \( Dg(p) = R^2 S^p \), then \( Dg(p)^{2} = S^p \circ (R^2) \circ S^p = (R^2) \circ S^p \circ R \). Since \( S^p \) is symmetric positive definite, the closest orthogonal transform to \( Dg(p) \) is \( (R^2)^p \), which implies the second statement of the lemma.

Proof of Proposition 6. Let us assume \( v \) is not singular at \( p \), and let \( w \) be one of the \( N \) vectors of \( v(p) \). Since \( v \) is stationary (as a \( N \)-symmetry field) for \( R^2 \), then \( R^2 w \) must also be one of the vectors of \( v(p) \), i.e. \( w \) and \( R^2 w \) must form an angle of \( 2k\pi/N \) for some integer \( k = 0, \ldots, N - 1 \). Since \( R^2 \) is a pure reflection about an axis \( t \), this may happen only if \( w \) and \( t \) form an angle of \( k\pi/N \).

References